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## On $k$ -decomposability of positive maps

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### Abstract

We extend the theory of decomposable maps by giving a detailed description of  $k$ -positive maps. A relation between transposition and modular theory is established. The structure of positive maps in terms of modular theory (the generalized Tomita–Takesaki scheme) is examined.

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## 1. Definitions, notations and statement of the problem

For any  $C^*$ -algebra  $A$  let  $A^+$  denote the set of all positive elements in  $A$ . A *state* on a unital  $C^*$ -algebra  $A$  is a linear functional  $\omega : A \rightarrow \mathbb{C}$ , such that  $\omega(a) \geq 0$  for every  $a \in A^+$  and  $\omega(\mathbb{1}) = 1$ , where  $\mathbb{1}$  is the unit of  $A$ . We will denote the set of all states on  $A$  by  $\mathcal{S}(A)$ . For any Hilbert space  $H$  we denote the set of all bounded linear operators on  $H$  by  $\mathcal{B}(H)$ .

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A linear map  $\varphi : A \rightarrow B$  between  $C^*$ -algebras is called *positive* if  $\varphi(A^+) \subset B^+$ . For  $k \in \mathbb{N}$  we consider a map  $\varphi_k : M_k(A) \rightarrow M_k(B)$ , where  $M_k(A)$  and  $M_k(B)$  are the algebras of  $k \times k$  matrices with coefficients from  $A$  and  $B$ , respectively, and  $\varphi_k([a_{ij}]) = [\varphi(a_{ij})]$ . We say that  $\varphi$  is *k-positive* if the map  $\varphi_k$  is positive. The map  $\varphi$  is said to be *completely positive* when it is *k-positive* for every  $k \in \mathbb{N}$ .

A *Jordan morphism* between  $C^*$ -algebras  $A$  and  $B$  is a linear map  $\mathfrak{J} : A \rightarrow B$  which respects the Jordan structures of algebras  $A$  and  $B$ , i.e.  $\mathfrak{J}(ab + ba) = \mathfrak{J}(a)\mathfrak{J}(b) + \mathfrak{J}(b)\mathfrak{J}(a)$  for every  $a, b \in A$ . Let us recall that every Jordan morphism is a positive map but it need not be a completely positive one (in fact it need not even be 2-positive). It is commonly known [25] that every Jordan morphism  $\mathfrak{J} : A \rightarrow \mathcal{B}(H)$  is a sum of an  $*$ -morphism and an  $*$ -antimorphism.

The Stinespring theorem states that every completely positive map  $\varphi : A \rightarrow \mathcal{B}(H)$  has the form  $\varphi(a) = W^*\pi(a)W$ , where  $\pi$  is an  $*$ -representation of  $A$  on some Hilbert space  $K$ , and  $W$  is a bounded operator from  $H$  to  $K$ .

Following Størmer [24], we say that a map  $\varphi : A \rightarrow \mathcal{B}(H)$  is *decomposable* if there are a Hilbert space  $K$ , a Jordan morphism  $\mathfrak{J} : A \rightarrow \mathcal{B}(K)$ , and a bounded linear operator  $W$  from  $H$  to  $K$ , such that  $\varphi(a) = W^*\mathfrak{J}(a)W$  for every  $a \in A$ .

Let  $(e_i)$  be a fixed orthonormal basis in some Hilbert space  $H$ . Define a conjugation  $J_c$  associated with this basis by the formula

$$J_c \left( \sum_i \lambda_i e_i \right) = \sum_i \overline{\lambda_i} e_i.$$

The map  $J_c$  has the following properties: (i)  $J_c$  is an antilinear isomorphism of  $H$ ; (ii)  $J_c^2 = \mathbb{I}$ ; (iii)  $\langle J_c \xi, J_c \eta \rangle = \langle \eta, \xi \rangle$  for every  $\xi, \eta \in H$ ; and (iv) the map  $a \mapsto J_c a J_c$  is an automorphism of the algebra  $\mathcal{B}(H)$ . For every  $a \in \mathcal{B}(H)$  we denote by  $a^t$  the element  $J_c a^* J_c$  and we call it a *transposition* of the element  $a$ . From the above properties (i)–(iv) it follows that the *transposition map*  $a \mapsto a^t$  is a linear  $*$ -antiautomorphism of  $\mathcal{B}(H)$ .

We say that a linear map  $\varphi : A \rightarrow \mathcal{B}(H)$  is *k-copositive* (resp., *completely copositive*) if the map  $a \mapsto \varphi(a)^t$  is *k-positive* (resp., *completely positive*). The following theorem characterizes decomposable maps in the spirit of Stinespring's theorem:

**Theorem 1.1** (Størmer [27]). *Let  $\varphi : A \rightarrow \mathcal{B}(H)$  be a linear map. Then the following conditions are equivalent:*

- (i)  $\varphi$  is decomposable;
- (ii) for every natural number  $k$  and for every matrix  $[a_{ij}] \in M_k(A)$ , such that both  $[a_{ij}]$  and  $[a_{ji}]$  belong to  $M_k(A)^+$  the matrix  $[\varphi(a_{ij})]$  is in  $M_k(\mathcal{B}(H))^+$ ; and
- (iii) there are maps  $\varphi_1, \varphi_2 : A \rightarrow \mathcal{B}(H)$ , such that  $\varphi_1$  is completely positive and  $\varphi_2$  is completely copositive, with  $\varphi = \varphi_1 + \varphi_2$ .

In spite of enormous efforts, the classification of decomposable maps is still not complete even in the case when  $A$  and  $H$  are finite dimensional, i.e.  $A = \mathcal{B}(\mathbb{C}^m)$  and  $H = \mathbb{C}^n$ . The most important step was done by Størmer [27], Choi [6,7] and Woronowicz [31]. Størmer and Woronowicz proved that if  $m = n = 2$  or  $m = 2, n = 3$  then every positive map is

decomposable. The first examples of non-decomposable maps were given by Choi (in the case  $m=n=3$ ) and Woronowicz (in the case  $m=2, n=4$ ). It seems that the main difficulty in carrying out the classification of positive maps is the question of the canonical form of non-decomposable maps. As far as we know there are only special examples of maps from that class which are scattered across the literature [7,10,11,14,15,23,26,31]. In fact, it seems that in the infinite-dimensional case all known examples of non-decomposable maps rely on the deep structure theory of the underlying algebras. (See for example [26].) On the other hand, it seems that very general positive maps (so not of the CP class) and hence possibly non-decomposable ones are crucial for an analysis of non-trivial quantum correlations, i.e. for an analysis of genuine quantum maps [12,18,19,22,30]. Having this motivation in mind we wish to present a step toward a canonical prescription for the construction of decomposable and non-decomposable maps. Namely, we study the notion of  $k$ -decomposability and prove an analog of Theorem 1.1. The basic strategy of the paper is to employ two dual pictures: one given in terms of operator algebras while the second one will use the space of states. Thus, it can be said that we are using the equivalence of the Schrödinger and Heisenberg pictures in the sense of Kadison [13], Connes [8] and Alfsen and Shultz [1].

The paper is organized as follows: in Section 2 we recall the techniques used in [20] and compare it with results from [16]. In Section 3, we formulate our main result concerning the notion of  $k$ -decomposability. Section 4 is devoted to a modification of the Tomita–Takesaki theory. Section 5, based on the previous section, presents a description of  $k$ -decomposability at the Hilbert-space level. Section 6 provides new results on partial transposition which are used to complete the description of  $k$ -decomposability.

We point out that although much of Sections 3–6 may appear to be familiar, most of the results contained therein are in fact new in the sense of having been proved in the  $k$ -positive context (as opposed to the completely positive context) for the first time. The extension of the cycle of ideas in Section 3 to this more general context is also more than just a simple exercise, with ad hoc techniques often being required to achieve the extension.

## 2. Dual construction

Most of the results in this section are known, but since they are a very necessary foundation for all that follows, we choose to state these results explicitly for the sake of continuity and clarity. Let us recall the construction of Choi [6] (see also [20]) which establishes a one-to-one correspondence between elements of  $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$  and linear maps from  $\mathcal{B}(\mathbb{C}^m)$  to  $\mathcal{B}(\mathbb{C}^n)$ . Fix some orthonormal basis  $e_1, e_2, \dots, e_m$  (resp.,  $f_1, f_2, \dots, f_n$ ) in  $\mathbb{C}^m$  (resp.,  $\mathbb{C}^n$ ) and by  $E_{ij}$  (resp.,  $F_{kl}$ ) denote the matrix units in  $\mathcal{B}(\mathbb{C}^m)$  (resp.,  $\mathcal{B}(\mathbb{C}^n)$ ). For any  $x \in \mathbb{C}^m$  define the linear operator  $V_x : \mathbb{C}^n \rightarrow \mathbb{C}^m \otimes \mathbb{C}^n$  by  $V_x y = x \otimes y$  where  $y \in \mathbb{C}^n$ . For simplicity, we write  $V_i$  instead of  $V_{e_i}$  for every  $i = 1, \dots, m$ . Observe that for any  $h \in \mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ , we have

$$h = \sum_{i,j=1}^m E_{ij} \otimes V_i^* h V_j. \quad (2.1)$$

Consequently, for every  $h$  one can define the map  $\varphi_h : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  by

$$\varphi_h(E_{ij}) = V_i^* h V_j, \quad i, j = 1, 2, \dots, m.$$

On the other hand, following (2.1) given a linear map  $\varphi : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  one can reconstruct  $h$  by the formula

$$h = \sum_{i,j=1}^m E_{ij} \otimes \varphi(E_{ij}) = (\text{id} \otimes \varphi) \left( \sum_{i,j=1}^m E_{ij} \otimes E_{ij} \right). \quad (2.2)$$

We summarize the main properties of this correspondence in the following:

**Theorem 2.1.** *Let  $h^* = h$ . Then [Choi [7] and Majewski and Marciniak [22]]*

(i) *the map  $\varphi_h$  is completely positive if and only if  $h$  is a positive operator, i.e.*

$$\langle z, h z \rangle \geq 0$$

*for every  $z \in \mathbb{C}^m \otimes \mathbb{C}^n$ ;*

(ii) *the map  $\varphi_h$  is positive if and only if*

$$\langle x \otimes y, h(x \otimes y) \rangle \geq 0 \quad (2.3)$$

*for every  $x \in \mathbb{C}^m$  and  $y \in \mathbb{C}^n$ ; and*

(iii) *the map  $\varphi_h$  is decomposable if and only if  $\omega(h) \geq 0$  for each state  $\omega$  on  $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ , such that  $\omega \circ (t \otimes \text{id})$  is also a state.*

If the operator  $h$  fulfills property (2.3) we will call it a *block-positive* operator.

In this section, we compare Theorem 2.1 with the results presented in [16]. For the reader's convenience we recall the main theorem from this paper.

**Theorem 2.2.** *A linear map  $\varphi : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  is positive if and only if it is of the form*

$$\varphi(a) = \sum_{k,l=1}^n \text{Tr}(a g_{lk}) F_{kl}, \quad a \in \mathcal{B}(\mathbb{C}^m),$$

where  $g_{kl} \in \mathcal{B}(\mathbb{C}^m)$ ,  $k, l = 1, \dots, n$ , satisfy the following condition: for every  $x \in \mathbb{C}^m$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$

$$\sum_{k,l=1}^n \lambda_k \bar{\lambda}_l \langle x, g_{kl} x \rangle \geq 0. \quad (2.4)$$

In fact condition (2.4) coincides with (2.3).

**Proposition 2.3.** *Let  $A \in \mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ . Then the following conditions are equivalent:*

(i) *for every  $x \in \mathbb{C}^m$  and  $y \in \mathbb{C}^n$*

$$\langle x \otimes y, Ax \otimes y \rangle \geq 0$$

(ii) *for every  $x \in \mathbb{C}^m$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$*

$$\sum_{k,l=1}^n \lambda_k \overline{\lambda_l} \langle x, A_{kl}x \rangle \geq 0,$$

*where  $A_{kl}$  are unique elements of  $\mathcal{B}(\mathbb{C}^m)$ , such that,  $A = \sum_{k,l} A_{kl} \otimes F_{kl}$ ; and*

(iii) *for every  $y \in \mathbb{C}^n$  and  $\mu_1, \dots, \mu_m \in \mathbb{C}$*

$$\sum_{i,j=1}^m \mu_i \overline{\mu_j} \langle y, A'_{ij}y \rangle \geq 0,$$

*where  $A'_{ij}$  are unique elements of  $\mathcal{B}(\mathbb{C}^n)$ , such that,  $A = \sum_{i,j} E_{ij} \otimes A'_{ij}$ .*

**Proof.** (i)  $\iff$  (ii) Let the  $\lambda$ 's be coefficients of the expansion of  $y$  in the basis  $\{f_k\}$ , i.e.  $y = \sum_s \lambda_s f_s$ . Then we have

$$\begin{aligned} \langle x \otimes y, Ax \otimes y \rangle &= \sum_{s,t} \lambda_s \overline{\lambda_t} \langle x \otimes f_t, Ax \otimes f_s \rangle \\ &= \sum_{s,t} \sum_{k,l} \lambda_s \overline{\lambda_t} \langle x, A_{kl}x \rangle \langle f_t, F_{kl}f_s \rangle = \sum_{k,l} \lambda_k \overline{\lambda_l} \langle x, A_{kl}x \rangle. \end{aligned}$$

This proves the equivalence.

(i)  $\iff$  (iii) This follows by the same method.  $\square$

The next proposition establishes the connection between the two constructions

**Proposition 2.4.** *Let  $\varphi : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  be a linear map. If*

$$g = \sum_{k,l} g_{kl} \otimes F_{kl},$$

*where  $\{g_{kl}\}$  are operators described in Theorem 2.2 and  $h$  is the operator defined in (2.2) then  $h = g^\dagger$ .*

**Proof.** Define the sesquilinear form  $(\cdot, \cdot)$  on  $\mathcal{B}(\mathbb{C}^m)$  by  $(a, b) = \text{Tr}(a^*b)$  for  $a, b \in \mathcal{B}(\mathbb{C}^m)$ . Then  $\mathcal{B}(\mathbb{C}^m)$  becomes a Hilbert space and  $\{E_{ij}\}$  forms an orthonormal basis. From the

definitions of  $h$  and  $g$  we get obtain

$$\begin{aligned} h &= \sum_{i,j} E_{ij} \otimes \varphi(E_{ij}) = \sum_{i,j} \sum_{k,l} \text{Tr}(E_{ij} B_{lk}) E_{ij} \otimes F_{kl} \\ &= \sum_{k,l} \left( \sum_{i,j} (E_{ji}, g_{lk}) E_{ji} \right)^t \otimes F_{kl} = \sum_{kl} g_{lk}^t \otimes F_{lk}^t = g^t. \quad \square \end{aligned}$$

### 3. $k$ -decomposability

The following theorem characterizes the  $k$ -positivity of a map  $\varphi$  in terms of the properties of operators  $g$  and  $h$  and constitutes a generalization of Theorems 2.1 and 2.2.

**Theorem 3.1.** *Let  $\varphi : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  be a linear map. Then the following conditions are equivalent:*

- (i)  $\varphi$  is  $k$ -positive;
- (ii) for every  $y_1, \dots, y_m \in \mathbb{C}^n$ , such that  $\dim \text{span}\{y_1, \dots, y_m\} \leq k$ , we have

$$\sum_{i,j=1}^n \langle y_j, h_{ij} y_i \rangle \geq 0,$$

where  $h_{ij} \in \mathcal{B}(\mathbb{C}^n)$  are such that  $h = \sum_{i,j} E_{ij} \otimes h_{ij}$ , i.e.  $h_{ij} = \varphi(E_{ij})$ ; and

- (iii) for every  $x_1, \dots, x_n \in \mathbb{C}^m$ , such that  $\dim \text{span}\{x_1, \dots, x_n\} \leq k$ , we have

$$\sum_{k,l=1}^n \langle x_k, g_{kl}^t x_l \rangle \geq 0.$$

**Proof.** (i)  $\iff$  (ii) Denote by  $\{k_\alpha\}_{\alpha=1}^k$  and  $\{K_{\alpha\beta}\}_{\alpha,\beta=1}^k$  the standard orthonormal basis in  $\mathbb{C}^k$  and the standard system of matrix units in  $M_k$ , respectively. By Theorem 2.1 the map  $\varphi_k = \text{id} \otimes \varphi : M_k \otimes \mathcal{B}(\mathbb{C}^m) \rightarrow M_k \otimes \mathcal{B}(\mathbb{C}^n)$  is positive if and only if

$$\langle x^{(k)} \otimes y^{(k)}, h^{(k)} x^{(k)} \otimes y^{(k)} \rangle \geq 0$$

for every  $x^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^m$  and  $y^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^n$ , where

$$\begin{aligned} h^{(k)} &= \sum_{\alpha,\beta=1}^k \sum_{i,j=1}^m K_{\alpha\beta} \otimes E_{ij} \otimes \varphi_k(K_{\alpha\beta} \otimes E_{ij}) \\ &= \sum_{\alpha,\beta=1}^k \sum_{i,j=1}^m K_{\alpha\beta} \otimes E_{ij} \otimes K_{\alpha\beta} \otimes h_{ij}. \end{aligned}$$

Let  $x^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^m$  and  $y^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^n$ , and let  $x_1, \dots, x_k \in \mathbb{C}^m$ ,  $y_1, \dots, y_k \in \mathbb{C}^n$  be such that

$$x^{(k)} = \sum_{\rho} k_{\rho} \otimes x_{\rho}, \quad y^{(k)} = \sum_{\sigma} k_{\sigma} \otimes y_{\sigma}.$$

Then

$$\begin{aligned} & \langle x^{(k)} \otimes y^{(k)}, h^{(k)} x^{(k)} \otimes y^{(k)} \rangle \\ &= \sum_{\rho, \sigma, \rho', \sigma'} \langle k_{\rho} \otimes x_{\rho} \otimes k_{\sigma} \otimes y_{\sigma}, h^{(k)} k_{\rho'} \otimes x_{\rho'} \otimes k_{\sigma'} \otimes y_{\sigma'} \rangle \\ &= \sum_{\rho, \sigma, \rho', \sigma'} \sum_{\alpha, \beta} \sum_{i, j} \langle k_{\rho}, K_{\alpha\beta} k_{\rho'} \rangle \langle x_{\rho}, E_{ij} x_{\rho'} \rangle \langle k_{\sigma}, K_{\alpha\beta} k_{\sigma'} \rangle \langle y_{\sigma}, h_{ij} y_{\sigma'} \rangle \\ &= \sum_{\alpha, \beta} \sum_{i, j} \langle x_{\beta}, E_{ij} x_{\alpha} \rangle \langle y_{\beta}, h_{ij} y_{\alpha} \rangle \\ &= \sum_{\alpha, \beta} \sum_{i, j} \langle e_i, x_{\alpha} \rangle \langle x_{\beta}, e_j \rangle \langle y_{\beta}, h_{ij} y_{\alpha} \rangle \\ &= \sum_{i, j} \left\langle \sum_{\beta} \langle e_j, x_{\beta} \rangle y_{\beta}, h_{ij} \sum_{\alpha} \langle e_i, x_{\alpha} \rangle y_{\alpha} \right\rangle. \end{aligned}$$

Let  $y'_i = \sum_{\alpha} \langle e_i, x_{\alpha} \rangle y_{\alpha}$  for  $i = 1, \dots, m$ . Then, the equivalence is obvious.

(ii)  $\iff$  (iii) This is a consequence of the following equality:

$$\begin{aligned} & \sum_{i, j} \langle y_j, h_{ij} y_i \rangle \\ &= \sum_{i, j} \sum_{s, t} \langle e_s, E_{ij} e_t \rangle \langle y_s, h_{ij} y_t \rangle \\ &= \sum_{s, t} \left\langle e_s \otimes y_s, \left( \sum_{i, j} E_{ij} \otimes h_{ij} \right) e_t \otimes y_t \right\rangle = \sum_{s, t} \langle e_s \otimes y_s, h e_t \otimes y_t \rangle \\ &= \sum_{s, t} \langle e_s \otimes y_s, g^t e_t \otimes y_t \rangle = \sum_{s, t} \left\langle e_s \otimes y_s, \left( \sum_{k, l} g_{kl}^t \otimes F_{lk} \right) e_t \otimes y_t \right\rangle \\ &= \sum_{s, t} \sum_{k, l} \langle e_s, g_{kl}^t e_t \rangle \langle y_s, F_{lk} y_t \rangle \\ &= \sum_{k, l} \sum_{s, t} \sum_{p, r} \overline{\langle f_p, y_s \rangle} \langle f_r, y_t \rangle \langle e_s, g_{kl}^t e_t \rangle \langle f_p, F_{lk} f_r \rangle \\ &= \sum_{k, l} \sum_{s, t} \overline{\langle f_k, y_s \rangle} \langle f_l, y_t \rangle \langle e_s, g_{kl}^t e_t \rangle = \sum_{k, l} \left\langle \sum_s \langle f_k, y_s \rangle e_s, g_{kl}^t \sum_t \langle f_l, y_t \rangle e_t \right\rangle. \end{aligned}$$

Now, define  $x_k = \sum_s \langle f_k, y_s \rangle e_s$  for  $k = 1, \dots, n$ . The equivalence follows from the fact that

$$\dim \operatorname{span}\{x_1, \dots, x_n\} = \dim \operatorname{span}\{y_1, \dots, y_m\}. \quad \square$$

As a corollary we obtain the following:

**Theorem 3.2.** *Let  $\varphi : \mathcal{B}(\mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^n)$  be a linear map. Then the following conditions are equivalent:*

- (i)  $\varphi$  is  $k$ -copositive;
- (ii) for every  $y_1, \dots, y_m \in \mathbb{C}^n$  such that  $\dim \operatorname{span}\{y_1, \dots, y_m\} \leq k$  we have

$$\sum_{i,j=1}^n \langle y_i, h_{ij} y_j \rangle \geq 0;$$

- (iii) for every  $x_1, \dots, x_n \in \mathbb{C}^m$  such that  $\dim \operatorname{span}\{x_1, \dots, x_n\} \leq k$  we have

$$\sum_{k,l=1}^n \langle x_k, g_{kl} x_l \rangle \geq 0.$$

**Proof.** With  $t$  denoting the transposition map  $a \rightarrow a^t$ , we let  $h'$  and  $g'$  denote the operators corresponding to the map  $\varphi \circ t$  in the construction described in Theorems 2.1 and 2.2 for  $\varphi$ . Then, it is easy to show that  $h'_{ij} = h_{ji}$  for every  $i, j = 1, \dots, m$  and  $g'_{kl} = g_{kl}^t$  for  $k, l = 1, \dots, n$ . Thus, the theorem follows.  $\square$

Now, we can generalize this result to the general case. If  $H$  is a Hilbert space let  $\operatorname{Proj}_k(H) = \{p \in \mathcal{B}(H) : p^* = p = p^2, \operatorname{Tr} p \leq k\}$ . Then we have the following:

**Theorem 3.3.** *Let  $A$  be a  $C^*$ -algebra,  $H$  a Hilbert space (not necessarily finite dimensional) and  $\varphi : A \rightarrow \mathcal{B}(H)$  a linear map. Then the following conditions are equivalent:*

- (i)  $\varphi$  is  $k$ -positive;
- (ii) for every  $n \in \mathbb{N}$ , every set of vectors  $\xi_1, \xi_2, \dots, \xi_n \in H$  such that

$$\dim \operatorname{span}\{\xi_1, \xi_2, \dots, \xi_n\} \leq k$$

and every  $[a_{ij}] \in M_n(A)^+$ , we have

$$\sum_{i,j=1}^n \langle \xi_i, \varphi(a_{ij}) \xi_j \rangle \geq 0;$$

- (iii) for every  $p \in \operatorname{Proj}_k(H)$  the map  $A \ni a \mapsto p\varphi(a)p \in \mathcal{B}(H)$  is completely positive.

**Proof.** (i)  $\Rightarrow$  (iii) Observe that the map  $p\varphi p$  is  $k$ -positive as it is a composition of  $k$ -positive and completely positive maps. It maps  $A$  into  $p\mathcal{B}(H)p$ , but the latter subalgebra is isomorphic with  $M_d$  where  $d = \operatorname{Tr} p \leq k$ . By the theorem of Tomiyama [29] the  $k$ -positivity of  $p\varphi p$  implies its complete positivity.



(iii)  $\Rightarrow$  (ii) Let  $\xi_1, \xi_2, \dots, \xi_n \in H$  and  $\dim \operatorname{span}\{\xi_1, \xi_2, \dots, \xi_n\} \leq k$ . If  $p$  is a projection such that  $pH = \operatorname{span}\{\xi_1, \xi_2, \dots, \xi_n\}$ , then  $p \in \operatorname{Proj}_k(H)$  and hence  $p\varphi p$  is completely positive by assumption. So, for every  $[a_{ij}] \in M_n(A)^+$  we have

$$\sum_{i,j=1}^n \langle \xi_i, \varphi(a_{ij})\xi_j \rangle = \sum_{i,j=1}^n \langle p\xi_i, \varphi(a_{ij})p\xi_j \rangle = \sum_{i,j=1}^n \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle \geq 0.$$

(ii)  $\Rightarrow$  (i) Let  $[a_{ij}] \in M_k(A)^+$ . Then for every  $\xi_1, \xi_2, \dots, \xi_k \in H$  we have

$$\sum_{i,j=1}^k \langle \xi_i, \varphi(a_{ij})\xi_j \rangle \geq 0.$$

This condition is equivalent to the positivity of the matrix  $[\varphi(a_{ij})]$  in  $M_k(\mathcal{B}(H))$ , which implies that  $\varphi$  is  $k$ -positive.  $\square$

**Corollary 3.4.** *A map  $\varphi : A \rightarrow \mathcal{B}(H)$  is completely positive if and only if  $p\varphi p$  is completely positive for every finite-dimensional projector in  $\mathcal{B}(H)$ .*

Now, we are ready to study the notion of  $k$ -decomposability.

**Definition 3.5.** Let  $\varphi : A \rightarrow \mathcal{B}(H)$  be a linear map.

- (1) We say that  $\varphi$  is  *$k$ -decomposable* if there are maps  $\varphi_1, \varphi_2 : A \rightarrow \mathcal{B}(H)$ , such that  $\varphi_1$  is  $k$ -positive,  $\varphi_2$  is  $k$ -copositive and  $\varphi = \varphi_1 + \varphi_2$ .
- (2) We say that  $\varphi$  is *weakly  $k$ -decomposable* if there is a  $C^*$ -algebra  $E$ , a unital Jordan morphism  $\mathfrak{J} : A \rightarrow E$ , and a positive map  $\psi : E \rightarrow \mathcal{B}(H)$ , such that  $\psi|_{\mathfrak{J}(A)}$  is  $k$ -positive and  $\varphi = \psi \circ \mathfrak{J}$ .

**Theorem 3.6.** *For any linear map  $\varphi : A \rightarrow \mathcal{B}(H)$  consider the following conditions:*

- (D<sub>k</sub>)  $\varphi$  is  $k$ -decomposable;
- (W<sub>k</sub>)  $\varphi$  is weakly  $k$ -decomposable;
- (S<sub>k</sub>) for every matrix  $[a_{ij}] \in M_k(A)$  such that both  $[a_{ij}]$  and  $[a_{ji}]$  are in  $M_k(A)^+$  the matrix  $[\varphi(a_{ij})]$  is positive in  $M_k(\mathcal{B}(H))$ ;
- (P<sub>k</sub>) for every  $p \in \operatorname{Proj}_k(H)$  the map  $p\varphi p$  is decomposable.

*Then we have the following implications: (D<sub>k</sub>)  $\Rightarrow$  (W<sub>k</sub>)  $\Leftrightarrow$  (P<sub>k</sub>)  $\Leftrightarrow$  (S<sub>k</sub>).*

**Proof.** (D<sub>k</sub>)  $\Rightarrow$  (P<sub>k</sub>) If  $\varphi = \varphi_1 + \varphi_2$  with  $\varphi_1$  is  $k$ -positive and  $\varphi_2$   $k$ -copositive, then  $p\varphi p = p\varphi_1 p + p\varphi_2 p$ . From Theorem 3.3  $p\varphi_1 p$  is a completely positive map. Observe that  $p^\dagger \in \operatorname{Proj}_k(H)$  for every  $p \in \operatorname{Proj}_k(H)$ . Hence  $(p\varphi_2 p)^\dagger = p^\dagger \varphi_2^\dagger p^\dagger$  and  $(p\varphi_2 p)^\dagger$  is

completely positive. Thus,  $p\varphi p$  is a sum of a completely positive and completely copositive map, and hence  $p\varphi p$  is decomposable.

$(P_k) \Rightarrow (S_k)$  Let  $[a_{ij}] \in M_k(A)$  be such that  $[a_{ij}], [a_{ji}] \in M_k(A)^+$ . Suppose that  $\xi_1, \xi_2, \dots, \xi_k \in H$  and that  $p$  is a projector on  $H$ , such that  $pH = \text{span}\{\xi_1, \xi_2, \dots, \xi_k\}$ . Then

$$\sum_{i,j=1}^k \langle \xi_i, \varphi(a_{ij})\xi_j \rangle = \sum_{i,j=1}^k \langle p\xi_i, \varphi(a_{ij})p\xi_j \rangle = \sum_{i,j=1}^k \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle \geq 0,$$

where in the last inequality we have used the fact that the matrix  $[p\varphi(a_{ij})p]$  is positive by the theorem of Størmer. Hence the matrix  $[\varphi(a_{ij})]$  is positive.

$(S_k) \Rightarrow (P_k)$  Let  $p \in \text{Proj}_k(H)$  and  $d = \text{Tr } p$ . One should show that for every  $n \in \mathbb{N}$  and every matrix  $[a_{ij}] \in M_n(A)$ , such that  $[a_{ij}], [a_{ji}] \in M_n(A)^+$  the matrix  $[p\varphi(a_{ij})p]$  is also positive. To this end we will show that for any vector  $\xi_1, \xi_2, \dots, \xi_n$  the inequality

$$\sum_{i,j=1}^n \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle \geq 0 \quad (3.5)$$

holds. If  $n \leq k$  then we define vectors  $\eta_1, \eta_2, \dots, \eta_k$

$$\eta_i = \begin{cases} p\xi_i & \text{for } 1 \leq i \leq n, \\ 0 & \text{for } n < i \leq k \end{cases}$$

and a matrix  $[b_{ij}] \in M_k(A)$

$$b_{ij} = \begin{cases} a_{ij} & \text{for } 1 \leq i, j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously both matrices  $[b_{ij}]$  and  $[b_{ji}]$  are positive in  $M_k(A)$ . Thus

$$\sum_{i,j=1}^n \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle = \sum_{i,j=1}^k \langle \eta_i, \varphi(b_{ij})\eta_j \rangle \geq 0$$

by assumption. Now, let us assume that  $n = k + 1$ . Define  $\eta_i = p\xi_i$  for  $i = 1, 2, \dots, k + 1$ . As  $\dim \text{span}\{\eta_1, \eta_2, \dots, \eta_{k+1}\} \leq k$  then at least one of vectors  $\eta_1, \eta_2, \dots, \eta_{k+1}$ , say  $\eta_{k+1}$ ,

is a linear combination of the others, i.e.  $\eta_{k+1} = \sum_{i=1}^k \alpha_i \eta_i$ . Then

$$\begin{aligned}
 & \sum_{i,j=1}^{k+1} \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle \\
 &= \sum_{i,j=1}^{k+1} \langle \eta_i, \varphi(a_{ij})\eta_j \rangle \\
 &= \sum_{i,j=1}^k \langle \eta_i, \varphi(a_{ij})\eta_j \rangle + \sum_{i=1}^k \langle \eta_i, \varphi(a_{i,k+1})\eta_{k+1} \rangle \\
 &\quad + \sum_{j=1}^k \langle \eta_{k+1}, \varphi(a_{k+1,j})\eta_j \rangle + \langle \eta_{k+1}, \varphi(a_{k+1,k+1})\eta_{k+1} \rangle \\
 &= \sum_{i,j=1}^k \langle \eta_i, \varphi(a_{ij})\eta_j \rangle + \sum_{i,j=1}^k \langle \eta_i, \alpha_j \varphi(a_{i,k+1})\eta_j \rangle \\
 &\quad + \sum_{i,j=1}^k \langle \alpha_i \eta_i, \varphi(a_{k+1,j})\eta_j \rangle + \sum_{i,j=1}^k \langle \alpha_i \eta_i, \alpha_j \varphi(a_{k+1,k+1})\eta_j \rangle \\
 &= \sum_{i,j=1}^k \langle \eta_i, [\varphi(a_{ij}) + \alpha_j \varphi(a_{i,k+1}) + \overline{\alpha_i} \varphi(a_{k+1,j}) \\
 &\quad + \overline{\alpha_i} \alpha_j \varphi(a_{k+1,k+1})] \eta_j \rangle \\
 &= \sum_{i,j=1}^k \langle \eta_i, \varphi(b_{ij})\eta_j \rangle,
 \end{aligned}$$

where  $b_{ij} = a_{ij} + \alpha_j a_{i,k+1} + \overline{\alpha_i} a_{k+1,j} + \overline{\alpha_i} \alpha_j a_{k+1,k+1}$  for  $i, j = 1, 2, \dots, k$ . The fact that both matrices  $[b_{ij}]$  and  $[b_{ji}]$  are positive in  $M_k(A)$  follows from the following matrix equality:

$$\begin{aligned}
 & \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & 0 \\ b_{21} & b_{22} & \cdots & b_{2k} & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ b_{k1} & b_{k2} & \cdots & b_{kk} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & \cdots & 0 & \overline{\alpha_1} \\ 0 & 1 & \cdots & 0 & \overline{\alpha_2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & \overline{\alpha_k} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} [a_{ij}] \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k & 0 \end{bmatrix}.
 \end{aligned}$$

Hence, by assumption inequality (3.5) holds. We may continue the proof for larger  $n$  by a similar inductive argument.

$(W_k) \Leftrightarrow (S_k)$  We follow the proof of the theorem in [27]. For the reader's convenience we describe Størmer's argument:

$(W_k) \Rightarrow (S_k)$  If  $\mathfrak{J}$  is an  $*$ -homomorphism (resp.,  $*$ -antihomomorphism) and  $[a_{ij}]$  (resp.,  $[a_{ji}]$ ) is in  $M_k(A)^+$  then  $[\mathfrak{J}(a_{ij})]$  belongs to  $M_k(E)^+$ . Since every Jordan morphism is a sum of an  $*$ -homomorphism and an  $*$ -antimorphism, if both  $[a_{ij}]$  and  $[a_{ji}]$  belong to  $M_k(A)^+$  then  $[\mathfrak{J}(a_{ij})] \in M_k(\mathcal{B}(H))^+$ . Applying  $\psi$  now yields the fact that  $[\varphi(a_{ij})] \in M_k(\mathcal{B}(H))^+$ .

$(S_k) \Rightarrow (W_k)$  Assume that  $A \subset \mathcal{B}(L)$  for some Hilbert space  $L$ . Let

$$V = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{t'} \end{bmatrix} \in M_2(\mathcal{B}(L)) : a \in A \right\},$$

where  $t'$  is the transposition map with respect to some orthonormal basis in  $L$ . Then  $V$  is a self-adjoint subspace of  $M_2(\mathcal{B}(L))$  containing the identity. One can observe that both  $[a_{ij}]$  and  $[a_{ji}]$  belong to  $M_k(A)^+$  if and only if

$$\begin{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11}^{t'} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1k} & 0 \\ 0 & a_{1k}^{t'} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} a_{k1} & 0 \\ 0 & a_{k1}^{t'} \end{bmatrix} & \cdots & \begin{bmatrix} a_{kk} & 0 \\ 0 & a_{kk}^{t'} \end{bmatrix} \end{bmatrix} \in M_k(V)^+.$$

Thus, the map  $\psi : V \rightarrow \mathcal{B}(H)$  defined by

$$\psi \left( \begin{bmatrix} a & 0 \\ 0 & a^{t'} \end{bmatrix} \right) = \varphi(a) \tag{3.6}$$

is  $k$ -positive. Now, take  $E = M_2(\mathcal{B}(L))$  and define the Jordan morphism  $\mathfrak{J}: A \rightarrow M_2(\mathcal{B}(L))$  by

$$\mathfrak{J}(a) = \begin{bmatrix} a & 0 \\ 0 & a^{t'} \end{bmatrix}$$

to prove the statement.  $\square$

We end this section with the remark that it is still an open problem as to whether conditions  $(S_k)$ ,  $(P_k)$  and  $(W_k)$  are equivalent to  $k$ -decomposability. The main difficulty in proving the implication, say  $(S_k) \Rightarrow (D_k)$ , is to find a  $k$ -positive extension of the map  $\psi$  constructed in (3.6) to the whole algebra  $M_2(\mathcal{B}(L))$ . So, one should answer the following question:

Given a  $C^*$ -algebra  $A$  and a self-adjoint linear unital subspace  $S$ , find conditions for  $k$ -positive maps  $\psi : S \rightarrow \mathcal{B}(H)$  which guarantee the existence of a  $k$ -positive extension of  $\psi$  to the whole algebra  $A$ .

In other words, the analog of Arveson's extension theorem for completely positive maps should be proved ([4], see also [28]).

#### 4. Tomita–Takesaki scheme for transposition

Let  $H$  be a finite-dimensional (say  $n$ -dimensional) Hilbert space. We are concerned with a strongly positive map  $\varphi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ , i.e. a map such that  $\varphi(a^*a) \geq \varphi(a)^*\varphi(a)$  for every  $a \in \mathcal{B}(H)$  (also called a Schwarz map).

Define  $\omega \in \mathcal{B}(H)_{+,1}^*$  as  $\omega(a) = \text{Tr } \varrho a$ , where  $\varrho$  is an invertible density matrix, i.e. the state  $\omega$  is a faithful one. Denote by  $(H_\pi, \pi, \Omega)$  the GNS triple associated with  $(\mathcal{B}(H), \omega)$ . Then, one has

- $H_\pi$  is identified with  $\mathcal{B}(H)$ , where the inner product  $(\cdot, \cdot)$  is defined as  $(a, b) = \text{Tr } a^*b$ ,  $a, b \in \mathcal{B}(H)$ ;
- with the above identification:  $\Omega = \varrho^{1/2}$ ;
- $\pi(a)\Omega = a\Omega$ ;
- the modular conjugation  $J_m$  is the Hermitian involution:  $J_m a \varrho^{1/2} = \varrho^{1/2} a^*$ ;
- the modular operator  $\Delta$  is equal to the map  $\varrho \cdot \varrho^{-1}$  (see Eq. (1.4) in [3]).

We assume that  $\omega$  is invariant with respect to  $\varphi$ , i.e.  $\omega \circ \varphi = \omega$ . Now, let us consider the operator  $T_\varphi \in \mathcal{B}(H_\pi)$  defined by

$$T_\varphi(a\Omega) = \varphi(a)\Omega, \quad a \in \mathcal{B}(H).$$

Obviously  $T_\varphi$  is a contraction due to the strong positivity of  $\varphi$ .

As a next step let us define two conjugations:  $J_c$  on  $H$  and  $J$  on  $H_\pi$ . To this end we note that the eigenvectors  $\{x_i\}$  of  $\varrho = \sum_i \lambda_i |x_i\rangle\langle x_i|$  form an orthonormal basis in  $H$  (due to the faithfulness of  $\omega$ ). Hence we can define

$$J_c f = \sum_i \overline{\langle x_i, f \rangle} x_i \quad (4.7)$$

for every  $f \in H$ . Due to the fact that  $E_{ij} \equiv |x_i\rangle\langle x_j|$  form an orthonormal basis in  $H_\pi$  we can define a conjugation  $J$  on  $H_\pi$  in the similar way

$$J a \varrho^{1/2} = \sum_{ij} \overline{(E_{ij}, a \varrho^{1/2})} E_{ij}. \quad (4.8)$$

Obviously,  $J \varrho^{1/2} = \varrho^{1/2}$ .

Now let us define a transposition on  $\mathcal{B}(H)$  as the map  $a \mapsto a^t \equiv J_c a^* J_c$  where  $a \in \mathcal{B}(H)$ . We will denote by  $\tau$  the map induced on  $H_\pi$  by the transposition, i.e.

$$\tau a \varrho^{1/2} = a^t \varrho^{1/2}, \quad (4.9)$$

where  $a \in \mathcal{B}(H)$ . The main properties of the notions introduced above are the following:

**Proposition 4.1.** *Let  $a \in \mathcal{B}(H)$  and  $\xi \in H_\pi$ . Then*

$$a^t \xi = J a^* J \xi.$$

**Proof.** Let  $\xi = bq^{1/2}$  for some  $b \in \mathcal{B}(H)$ . Then we can perform the following calculations:

$$\begin{aligned}
 Ja^*Jbq^{1/2} &= \sum_{ij} \overline{(E_{ij}, a^*Jbq^{1/2})} E_{ij} = \sum_{ij} \sum_{kl} (E_{kl}, bq^{1/2}) \overline{(E_{ij}, a^*E_{kl})} E_{ij} \\
 &= \sum_{ijkl} \text{Tr}(E_{lk}bq^{1/2}) \overline{\text{Tr}(E_{ji}a^*E_{kl})} E_{ij} = \sum_{ijk} \text{Tr}(E_{jk}bq^{1/2}) \overline{\text{Tr}(E_{ki}a^*)} E_{ij} \\
 &= \sum_{ijk} \langle x_k, bq^{1/2}x_j \rangle \overline{\langle x_i, a^*x_k \rangle} E_{ij} = \sum_{ijk} \langle J_cbq^{1/2}x_j, x_k \rangle \langle x_k, ax_i \rangle E_{ij} \\
 &= \sum_{ij} \langle J_cbq^{1/2}x_j, ax_i \rangle E_{ij} = \sum_{ij} \langle a^*J_cbq^{1/2}x_j, x_i \rangle E_{ij} \\
 &= \sum_{ij} \langle x_i, J_ca^*J_cbq^{1/2}x_j \rangle E_{ij} = \sum_{ij} \langle x_i, a^tbq^{1/2}x_j \rangle E_{ij} \\
 &= \sum_{ij} \text{Tr}(E_{ji}a^tbq^{1/2}) E_{ij} = \sum_{ij} (E_{ij}, a^tbq^{1/2}) E_{ij} = a^tbq^{1/2}. \quad \square
 \end{aligned}$$

As a next step let us consider the modular conjugation  $J_m$  which has the form

$$J_m a q^{1/2} = (a q^{1/2})^* = q^{1/2} a^*. \quad (4.10)$$

Also define the unitary operator  $U$  on  $H_\pi$  by

$$U = \sum_{ij} |E_{ji}\rangle \langle E_{ij}|. \quad (4.11)$$

Clearly,  $UE_{ij} = E_{ji}$ . We have the following:

**Proposition 4.2.** *Let  $J$  and  $J_m$  be the conjugations introduced above and  $U$  be the unitary operator defined by (4.11). Then we have*

- (1)  $U^2 = \mathbb{I}$  and  $U = U^*$ ,
- (2)  $J = U J_m$ ; and
- (3)  $J, J_m$  and  $U$  mutually commute.

**Proof.** (1) We calculate

$$\sum_{ijmn} |E_{ij}\rangle \langle E_{ji}| |E_{mn}\rangle \langle E_{nm}| = \sum_{ijmn} \text{Tr}(E_{ij}E_{mn}) |E_{ij}\rangle \langle E_{nm}| = \sum_{ij} |E_{ij}\rangle \langle E_{ij}| = \mathbb{I}.$$

The rest is evident.

(2) Let  $b \in \mathcal{B}(H)$ . Then

$$\begin{aligned}
 U J_m b q^{1/2} &= U q^{1/2} b^* = \sum_{ij} (E_{ji}, q^{1/2} b^*) E_{ij} \\
 &= \sum_{ij} \text{Tr}(E_{ij} q^{1/2} b^*) E_{ij} = \sum_{ij} \langle x_j, q^{1/2} b^* x_i \rangle E_{ij} \\
 &= \sum_{ij} \overline{\langle x_i, b q^{1/2} x_j \rangle} E_{ij} = \sum_{ij} \overline{\text{Tr}(E_{ji} b q^{1/2})} E_{ij} \\
 &= \sum_{ij} \overline{(E_{ij}, b q^{1/2})} E_{ij} = J b q^{1/2}.
 \end{aligned}$$

(3)  $J$  is an involution, so by the previous point we have  $U J_m U J_m = \mathbb{I}$ . It is equivalent to the equality  $U J_m = J_m U$ . Hence, we obtain  $U J_m = J = J_m U$  and consequently  $U J = J_m = J U$  and  $J_m J = U = J J_m$  because both  $U$  and  $J_m$  are also involutions.  $\square$

Now, we are ready to describe a polar decomposition of the map  $\tau$ .

**Theorem 4.3.** *If  $\tau$  is the map introduced in (4.9), then*

$$\tau = U \Delta^{1/2}.$$

**Proof.** Let  $a \in \mathcal{B}(H)$ . Then by Propositions 4.1 and 4.2(2) we have

$$\tau a q^{1/2} = a^t q^{1/2} = J a^* J q^{1/2} = J J_m \Delta^{1/2} a q^{1/2} = U \Delta^{1/2} a q^{1/2}. \quad \square$$

Now we wish to prove some properties of  $U$  which are analogous to that of the modular conjugation  $J_m$ . To this end we firstly need the following:

**Lemma 4.4.**  *$J$  commutes with  $\Delta$ .*

**Proof.** Let  $a \in \mathcal{B}(H)$ . Then by Propositions 4.1, 4.2 and Theorem 4.3 we have

$$\begin{aligned}
 \Delta^{1/2} J a q^{1/2} &= \Delta^{1/2} J a J q^{1/2} = \Delta^{1/2} (a^*)^t q^{1/2} = U U \Delta^{1/2} (a^*)^t q^{1/2} \\
 &= U a^* q^{1/2} = U J J a^* J q^{1/2} = J U a^t q^{1/2} \\
 &= J U U \Delta^{1/2} a q^{1/2} = J \Delta^{1/2} a q^{1/2}.
 \end{aligned}$$

So,  $\Delta^{1/2} J = J \Delta^{1/2}$  and consequently  $\Delta J = \Delta^{1/2} J \Delta^{1/2} = J \Delta$ .  $\square$

We will also use (cf. [2])

$$V_\beta = \text{closure} \left\{ \Delta^\beta a q^{1/2} : a \geq 0, \beta \in \left[0, \frac{1}{2}\right] \right\}.$$

Clearly, each  $V_\beta$  is a pointed, generating cone in  $H_\pi$  and

$$V_\beta = \{\xi \in H_\pi : (\eta, \xi) \geq 0 \text{ for all } \eta \in V_{(1/2)-\beta}\}. \quad (4.12)$$

Recall that  $V_{1/4}$  is nothing but the natural cone  $\mathcal{P}$  associated with the pair  $(\pi(\mathcal{B}(H)), \Omega)$  (see [5, Proposition 2.5.26(1)]). Finally, let us define an automorphism  $\alpha$  on  $\mathcal{B}(H_\pi)$  by

$$\alpha(a) = UaU^*, \quad a \in \mathcal{B}(H_\pi). \quad (4.13)$$

Then we have

**Proposition 4.5.**

- (1)  $U\Delta = \Delta^{-1}U$ ;
- (2)  $\alpha$  maps  $\pi(\mathcal{B}(H))$  onto  $\pi(\mathcal{B}(H))'$ ; and
- (3) For every  $\beta \in [0, 1/2]$  the unitary  $U$  maps  $V_\beta$  onto  $V_{(1/2)-\beta}$ .

**Proof.** (1) By Proposition 4.2 and Lemma 4.4 we have

$$U\Delta = JJ_m\Delta = J\Delta^{-1}J_m = \Delta^{-1}JJ_m.$$

- (2) Let  $a, b \in \mathcal{B}(H)$  and  $\xi \in H_\pi$ . Then Propositions 4.1 and 4.2 imply

$$\begin{aligned} UaUb\xi &= JJ_m a J_m J b J J \xi = JJ_m a J_m (b^*)^\dagger J \xi = J(b^*)^\dagger J_m a J_m J \xi \\ &= J(b^*)^\dagger J J J_m a J_m J \xi = b J J_m a J_m J \xi = b U a U \xi \end{aligned}$$

and the proof is complete.

- (3) Let  $a, b \in \mathcal{B}(H)^+$ . Then by point (1) and Theorem 4.3 we have

$$\begin{aligned} (\Delta^\beta b q^{1/2}, U\Delta^\beta a q^{1/2}) &= (\Delta^\beta b q^{1/2}, \Delta^{(1/2)-\beta} U \Delta^{1/2} a q^{1/2}) \\ &= (\Delta^\beta b q^{1/2}, \Delta^{(1/2)-\beta} a^\dagger q^{1/2}). \end{aligned}$$

We recall that  $a \mapsto a^\dagger$  is a positive map on  $\mathcal{B}(H)$  so by (4.12) the last expression is non-negative. Hence  $UV_\beta \subset V_{(1/2)-\beta}$  for every  $\beta \in [0, \frac{1}{2}]$ . As  $U$  is an involution, we obtain  $V_{(1/2)-\beta} = U^2 V_{(1/2)-\beta} \subset UV_\beta$  and the proof is complete.  $\square$

**Corollary 4.6.**  $U\Delta^{1/2}$  and  $T_\phi U\Delta^{1/2}$  map  $V_0$  into itself.

Summarizing, this section establishes a close relationship between the Tomita–Takesaki scheme and transposition. Moreover, we have the following:

**Proposition 4.7.** Let  $\xi \mapsto \omega_\xi$  be the homeomorphism between the natural cone  $\mathcal{P}$  and the set of normal states on  $\pi(\mathcal{B}(H))$  described in [5, Theorem 2.5.31], i.e. such that

$$\omega_\xi(a) = (\xi, a\xi), \quad a \in \mathcal{B}(H).$$

For every state  $\omega$  define  $\omega^\tau(a) = \omega(a^\dagger)$  where  $a \in \mathcal{B}(H)$ . If  $\xi \in \mathcal{P}$  then the unique vector in  $\mathcal{P}$  mapped into the state  $\omega_\xi^\tau$  by the homeomorphism described above is equal to  $U\xi$ .



**Proof.** Let  $\xi = \Delta^{1/4} a \Omega$  for some  $a \in \mathcal{B}(H)^+$ . Then we have

$$\begin{aligned}
 (U \Delta^{1/4} a \Omega, x U \Delta^{1/4} a \Omega) &= (\Delta^{1/4} U \Delta^{1/2} a \Omega, x \Delta^{1/4} U \Delta^{1/2} a \Omega) \\
 &= (\Delta^{1/4} a^t \Omega, x \Delta^{1/4} a^t \Omega) \\
 &= (\Delta^{1/4} J a J \Omega, x \Delta^{1/4} J a J \Omega) \\
 &= (x^* J \Delta^{1/4} a \Omega, J \Delta^{1/4} a \Omega) \\
 &= (\Delta^{1/4} a \Omega, J x^* J \Delta^{1/4} a \Omega). \quad \square
 \end{aligned}$$

## 5. $k$ -decomposability at the Hilbert-space level

The results of Section 4 strongly suggest that a more complete theory of  $k$ -decomposable maps may be obtained in Hilbert-space terms. To examine this question we will study the description of positivity in the dual approach given in Section 3, i.e. we will be concerned with the approach on the Hilbert-space level.

Let  $\mathcal{M} \subset \mathcal{B}(H)$  be a concrete von Neumann algebra with a cyclic and separating vector  $\Omega$ . When used,  $\omega$  will denote the vector state  $\omega = (\Omega, \cdot \Omega)$ . The natural cone (modular operator) associated with  $(\mathcal{M}, \Omega)$  will be denoted by  $\mathcal{P}$  ( $\Delta$ , respectively).

We denote by  $\mathcal{P}_n$  the natural cone for  $(\mathcal{M} \otimes \mathcal{B}(\mathbb{C}^n), \omega \otimes \omega_0)$ , where  $\omega_0$  is a faithful state on  $\mathcal{B}(\mathbb{C}^n)$  (as an example of  $\omega_0$  one can take  $\frac{1}{n} \text{Tr}$ ). For the same algebra,  $\Delta_n = \Delta \otimes \Delta_0$  and  $J_n$ , being, respectively, the modular operator and modular conjugation for  $M_n(\mathcal{M})$ , are defined in terms of the vector  $\Omega_n = \Omega \otimes \Omega_0$  (i.e. in terms of the state  $\omega \otimes \omega_0$ ).

We will consider unital positive maps  $\varphi$  on  $\mathcal{M}$  which satisfy Detailed Balance II, i.e. there is another positive unital map  $\varphi^\beta$ , such that  $\omega(a^* \varphi(b)) = \omega(\varphi^\beta(a^*) b)$  (see [21]). Such maps induce bounded maps  $T_\varphi = T$  on  $H_\omega = H$  which commute strongly with  $\Delta$  and which satisfy  $T^*(\mathcal{P}) \subset \mathcal{P}$ . Now under the above assumptions [17, Lemma 4.10] assures us that this correspondence is actually 1-1. Partial transposition  $(\text{id} \otimes \tau)$  on  $M_n(\mathcal{M})$  also induces an operator at the Hilbert-space level, but for the sake of simplicity we will retain the notation  $(\text{id} \otimes \tau)$  for this operator where convenient.

In order to achieve the desired classification of positive maps we introduce the notion of the “transposed cone”  $\mathcal{P}_n^\tau = (\mathbb{I} \otimes U) \mathcal{P}_n$ , where  $\tau$  is a transposition on  $M_n(\mathbb{C})$  while the operator  $U$  was defined in the previous section (we have used the following identification: for the basis  $\{e_i\}_i$  in  $\mathbb{C}^n$  consisting of eigenvectors of  $\varrho_{\omega_0}$  ( $\omega_0(\cdot) = \text{Tr}\{\varrho_{\omega_0} \cdot\}$ ), we have the basis  $\{E_{ij} \equiv |e_i\rangle\langle e_j|\}_{ij}$  in the GNS Hilbert space associated with  $(\mathcal{B}(\mathbb{C}^n), \omega_0)$  with  $U$  defined in terms of that basis). Note that in the same basis one has the identification  $\mathcal{B}(\mathbb{C}^n)$  with  $M_n(\mathbb{C})$ .

Now the natural cone  $\mathcal{P}_n$  for  $\mathcal{M} \otimes \mathcal{B}(\mathbb{C}^n) = M_n(\mathcal{M})$  may be realized as

$$\mathcal{P}_n = \overline{\Delta_n^{1/4} \{[a_{ij}] \Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+\}}$$

(see for example [5, Proposition 2.5.26]). We observe

$$\begin{aligned} & \{(\mathbb{I} \otimes U)\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+\} \\ &= \{(\Delta_n^{1/4} \otimes U\Delta_0^{1/4}) \circ \sum a_{ij} \otimes E_{ij} : [a_{ij}] \in M_n(\mathcal{M})^+\} \\ &= \{(\Delta_n^{1/4} \otimes \Delta_0^{1/4} U\Delta_0^{1/2}) \circ \sum a_{ij} \otimes E_{ij} : [a_{ij}] \in M_n(\mathcal{M})^+\} \\ &= \{\Delta_n^{1/4}[a_{ji}]\Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+\}. \end{aligned}$$

Thus

$$\mathcal{P}_n^\tau = \overline{\Delta_n^{1/4}\{[a_{ji}]\Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+\}}.$$

The task of describing the transposed cone will be addressed more adequately in the next section.

**Lemma 5.1.** *The map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  is  $k$ -positive ( $k$ -copositive) if and only if  $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \mathcal{P}_n$  (respectively,  $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \mathcal{P}_n^\tau$ ) for every  $1 \leq n \leq k$ .*

**Proof.** To prove the  $k$ -positivity case it is enough to suitably adapt the proof of Majewski [17, Lemma 4.10], while to prove  $k$ -copositivity we observe that the “if” part of the hypothesis implies

$$0 \leq ((T_\varphi \otimes \mathbb{I})(\mathbb{I} \otimes U)\mathcal{P}_n, \mathcal{P}_n).$$

Thus

$$(\Delta_n^{1/4}([T(a_{ji})])\Omega_n, \Delta_n^{-1/4}([b_{kl}]^*[b_{kl}])\Omega_n) = ([T_\varphi(a_{ji})]\Omega_n, [b_{kl}]^*[b_{kl}]\Omega_n) \geq 0,$$

where  $[a_{ij}] \geq 0$  is in the algebra  $M_n(\mathcal{M})$ , and  $[b_{kl}]$  in its commutant. This implies that  $[T_\varphi(a_{ji})] \geq 0$  and the rest is again a suitable adaptation of the proof of Majewski [17, Lemma 4.10].  $\square$

**Lemma 5.2.** *For each  $n$ ,  $\mathcal{P}_n \cap \mathcal{P}_n^\tau$  and  $\overline{\text{co}}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$  are dual cones.*

**Proof.** For any  $X \subset H$  we denote  $X^d = \{\xi \in H : (\xi, \eta) \geq 0 \text{ for any } \eta \in X\}$ . To prove the lemma it is enough to observe that  $\mathcal{P}_n^d = \mathcal{P}_n$  and  $(\mathcal{P}_n^\tau)^d = \mathcal{P}_n^\tau$ .  $\square$

**Lemma 5.3.** *Let  $n$  be given. For any  $[a_{ij}] \in M_n(\mathcal{M})^+$ ,  $\Delta_n^{1/4}[a_{ij}]\Omega_n \in \mathcal{P}_n \cap \mathcal{P}_n^\tau$  implies  $[a_{ji}] \in M_n(\mathcal{M})^+$ .*

**Proof.** Let  $[a_{ij}] \in M_n(\mathcal{M})^+$  be given and assume that  $\Delta_n^{1/4}[a_{ij}]\Omega_n \in \mathcal{P}_n \cap \mathcal{P}_n^\tau$ . We observe that

$$\Delta_n^{1/4}[a_{ji}]\Omega_n = (\mathbb{I} \otimes U)\Delta_n^{1/4}[a_{ij}]\Omega_n \in (\mathbb{I} \otimes U)(\mathcal{P}_n \cap \mathcal{P}_n^\tau) = \mathcal{P}_n \cap \mathcal{P}_n^\tau \subset \mathcal{P}_n.$$

But then the self-duality of  $\mathcal{P}_n$  alongside [5, 2.5.26] will ensure that

$$0 \leq (\Delta_n^{1/4}[a_{ji}]\Omega_n, \Delta_n^{-1/4}[b_{ij}]\Omega_n) = ([a_{ji}]\Omega_n, [b_{ij}]\Omega_n)$$

for each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$ . We may now conclude from [9, 2.5.1] or [5, 2.3.19] that  $[a_{ji}] \geq 0$ , as required.  $\square$

**Corollary 5.4.** *In the finite-dimensional case  $\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ji}] \geq 0\} = \mathcal{P}_n \cap \mathcal{P}_n^\tau$ .*

**Proof.** First note that in this case  $\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0\} = \mathcal{P}_n$  (cf. [5, Proposition 2.5.26]). Now apply the previous lemma.  $\square$

Recall  $\Delta_n^{-1/4}$  maps  $\overline{\{[b_{ij}]\Omega_n : [b_{ij}] \in (M_n(\mathcal{M})')^+\}}$  densely into  $\mathcal{P}_n$  (see for example [5]). At least on a formal-level one may therefore by analogy with [5, 2.5.26, 2.5.27] expect to end up with a dense subset of the dual cone of  $\overline{\text{co}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)}$  (i.e. of  $\mathcal{P}_n \cap \mathcal{P}_n^\tau$ ) when applying  $\Delta_n^{1/4}$  to the set of all  $\alpha$ 's satisfying  $([b_{ij}]\Omega_n, \alpha) \geq 0$  and  $([b_{ij}]\Omega_n, (\mathbb{I} \otimes U)\alpha) \geq 0$  for each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$ . If true such a fact would then put one in a position to try and show that in general  $\mathcal{P}_n \cap \mathcal{P}_n^\tau = \overline{\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ji}] \geq 0\}}$ .

**Question.** Is it generally true that

$$\overline{\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ji}] \geq 0\}} = \mathcal{P}_n \cap \mathcal{P}_n^\tau?$$

In the light of the following result the answer to this becomes important in an attempt to generalize the finite case to the infinite-dimensional one.

**Theorem 5.5.** *In general the property  $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\text{co}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)}$  for each  $1 \leq n \leq k$  implies that  $\varphi$  is weakly  $k$ -decomposable in the sense that for each  $1 \leq n \leq k$ ,  $[\varphi(a_{ij})] \geq 0$  whenever  $[a_{ij}], [a_{ji}] \in M_n(\mathcal{M})^+$ .*

*If  $\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ji}] \geq 0\} = \mathcal{P}_n \cap \mathcal{P}_n^\tau$  for each  $1 \leq n \leq k$ , the converse implication also holds. In particular in the finite-dimensional case the two statements are equivalent. (Pending the answer to the aforementioned question, they may of course be equivalent in general.)*

**Proof.** Suppose that  $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\text{co}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)}$  for each  $1 \leq n \leq k$ . Given  $1 \leq n \leq k$  and  $[a_{ij}] \in M_n(\mathcal{M})$  it now follows from [5, Proposition 2.3.19] and the strong commutation of  $T_\varphi \otimes \mathbb{I}$  with  $\Delta_k$ , that  $[\varphi(a_{ij})] \geq 0$  if and only if

$$\begin{aligned} 0 &\leq (\varphi_n([a_{ij}])\Omega_n, [b_{ij}]\Omega_n) \\ &= ([a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*[b_{ij}]\Omega_n) \\ &= (\Delta_n^{1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n) \end{aligned}$$

for each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$ .

Now if  $[a_{ij}] \geq 0$  and  $[a_{ji}] \geq 0$ , then the fact that  $\text{id} \otimes \tau$  commutes strongly with  $\Delta_n$  surely ensures that  $\Delta_n^{1/4}[a_{ij}]\Omega_n \in \mathcal{P}_n \cap \mathcal{P}_n^\tau$ . Moreover, for any  $[b_{ij}] \in (M_n(\mathcal{M})')^+$ , [5, Proposition 2.5.26] alongside the hypothesis ensures that

$$(T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n \in \overline{\text{co}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)}.$$

In this case it therefore follows from the duality of  $\mathcal{P}_n \cap \mathcal{P}_n^\tau$  and  $\overline{\text{co}}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$  that  $0 \leq (\Delta_n^{1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^* \Delta_n^{-1/4}[b_{ij}]\Omega_n)$  for each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$ , and hence that  $[\varphi(a_{ij})] \geq 0$  as required.

For the converse suppose that  $\{\Delta_n^{1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ji}] \geq 0\} = \mathcal{P}_n \cap \mathcal{P}_n^\tau$  for each  $1 \leq n \leq k$  and that for each  $1 \leq n \leq k$  we have that  $[\varphi(a_{ij})] \geq 0$  whenever  $[a_{ij}], [a_{ji}] \in M_n(\mathcal{M})^+$ . To observe that  $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\text{co}}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$  for each  $1 \leq n \leq k$ , we need only show that  $(T_\varphi \otimes \mathbb{I})^*(\Delta_n^{-1/4}[b_{ij}]\Omega_n) \subset \overline{\text{co}}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$  for each  $1 \leq n \leq k$  and each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$  [5, Proposition 2.5.26]. To see that this is indeed the case, the duality of  $\mathcal{P}_n \cap \mathcal{P}_n^\tau$  and  $\overline{\text{co}}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$  ensures that we need only show that

$$0 \leq (\eta, (T_\varphi \otimes \mathbb{I})^* \Delta_n^{-1/4}[b_{ij}]\Omega_n)$$

for each  $\eta \in \mathcal{P}_n \cap \mathcal{P}_n^\tau$ . In the light of our assumption regarding  $\mathcal{P}_n \cap \mathcal{P}_n^\tau$ , this in turn means that we need to show that

$$\begin{aligned} 0 &\leq (\Delta_n^{1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^* \Delta_n^{-1/4}[b_{ij}]\Omega_n) \\ &= ([a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*[b_{ij}]\Omega_n) \\ &= (\varphi_n([a_{ij}])\Omega_n, [b_{ij}]\Omega_n) \end{aligned}$$

for each  $[b_{ij}] \in (M_n(\mathcal{M})')^+$  and each  $[a_{ij}] \in M_n(\mathcal{M})$  with  $[a_{ij}] \geq 0, [a_{ji}] \geq 0$ . Since by assumption  $[\varphi(a_{ij})] \geq 0$  whenever  $[a_{ij}] \geq 0, [a_{ji}] \geq 0$  ( $1 \leq n \leq k$ ), the claim therefore follows from [5, Proposition 2.3.19].  $\square$

## 6. Tomita–Takesaki approach for partial transposition

In order to obtain a more complete characterization of  $k$ -decomposable maps, one should describe elements of the cone  $\mathcal{P}_k \cap \mathcal{P}_k^\tau$  (cf. Theorem 5.5). In this section we formulate the general scheme for this description.

Suppose that  $A$  is a  $C^*$ -algebra equipped with a faithful state  $\omega_A$ . Let  $B = \mathcal{B}(K_B)$  for some Hilbert-space  $K_B$ ,  $\varrho$  be an invertible density matrix in  $\mathcal{B}(K)$  and  $\omega_B$  be a state on  $B$  such that  $\omega_B(b) = \text{Tr}(b\varrho)$  for  $b \in B$ . By  $(H, \pi, \Omega)$ ,  $(H_A, \pi_A, \Omega_A)$  and  $(H_B, \pi_B, \Omega_B)$  we denote the GNS representations of  $(A \otimes B, \omega_A \otimes \omega_B)$ ,  $(A, \omega_A)$  and  $(B, \omega_B)$ , respectively. We observe that we can make the following identifications:

- (1)  $H = H_A \otimes H_B$ ,
- (2)  $\pi = \pi_A \otimes \pi_B$ , and
- (3)  $\Omega = \Omega_A \otimes \Omega_B$ .

With these identifications we have  $J_m = J_A \otimes J_B$  and  $\Delta = \Delta_A \otimes \Delta_B$ , where  $J_m$ ,  $J_A$ , and  $J_B$  are modular conjugations and  $\Delta$ ,  $\Delta_A$ , and  $\Delta_B$  are modular operators for  $(\pi(A \otimes B)'', \Omega)$ ,  $(\pi_A(A)'', \Omega_A)$ ,  $(\pi_B(B)'', \Omega_B)$ , respectively. Since  $\Omega_A$  and  $\Omega_B$  are separating vectors, we will write  $a\Omega_A$  and  $b\Omega_B$  instead of  $\pi_A(a)\Omega_A$  and  $\pi_B(b)\Omega_B$  for  $a \in A$  and  $b \in B$ .

The natural cone  $\mathcal{P}$  for  $(\pi(A \otimes B)'', \Omega)$  is defined (see [5] or [2]) as the closure of the set

$$\left\{ \left( \sum_{k=1}^n a_k \otimes b_k \right) j_m \left( \sum_{l=1}^n a_l \otimes b_l \right) \Omega : n \in \mathbb{N}, a_1, \dots, a_n \in A, b_1, \dots, b_n \in B \right\},$$

where  $j_m(\cdot) = J_m \cdot J_m$  is the modular morphism on  $\pi(A \otimes B)'' = \pi_A(A)'' \otimes \pi_B(B)''$ .

Recall (see Section 4) that  $H_B$  is the closure of the set  $\{bq^{1/2} : b \in B\}$  and  $\Omega_B$  can be identified with  $q^{1/2}$ . Let  $U_B$  be the unitary operator on  $H_B$  described in Section 4. Then we have the following:

**Lemma 6.1.** *With  $\alpha(\cdot) = U_B(\cdot)U_B^*$ , the set  $(\mathbb{I} \otimes U_B)\mathcal{P}$  is the closure of the set*

$$\left\{ \left( \sum_{k=1}^n a_k \otimes \alpha(b_k) \right) j_m \left( \sum_{l=1}^n a_l \otimes \alpha(b_l) \right) \Omega : n \in \mathbb{N}, \begin{matrix} a_1, \dots, a_n \in A \\ b_1, \dots, b_n \in B \end{matrix} \right\}.$$

**Proof.** Using the Tomita–Takesaki approach one has

$$\begin{aligned} & (\mathbb{I} \otimes U_B) \left( \sum_k a_k \otimes b_k \right) j_m \left( \sum_l a_l \otimes b_l \right) \Omega \\ &= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k J_B b_l J_B \Omega_B \\ &= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k U_B U_B J_B b_l \Omega_B \\ &= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k U_B J_B U_B b_l U_B J_B \Omega_B \\ &= \left( \sum_k a_k \otimes \alpha(b_k) \right) j_m \left( \sum_l a_l \otimes \alpha(b_l) \right) \Omega. \end{aligned}$$

In the third equality we used the fact that  $U_B$  commutes with  $J_B$ .  $\square$

This leads us to the following:

**Theorem 6.2.** *Suppose that  $K$  is a finite-dimensional Hilbert space. Then  $(\mathbb{I} \otimes U_B)\mathcal{P} = \mathcal{P}'$  where  $\mathcal{P}'$  is the natural cone associated with  $(\pi_A(A) \otimes \pi_B(B)', \Omega)$ .*

**Proof.** We just proved that  $(\mathbb{I} \otimes U_B)\mathcal{P}$  is the closure of the set

$$\left\{ \left( \sum_{k=1}^n a_k \otimes \alpha(b_k) \right) \times j_m \left( \sum_{l=1}^n a_l \otimes \alpha(b_l) \right) \Omega : n \in \mathbb{N}, a_1, \dots, a_n \in A, b_1, \dots, b_n \in B \right\}.$$

By Proposition 4.5(2)  $\alpha$  maps  $\pi_B(B)''$  onto  $\pi_B(B)'$ , so the assertion is obvious.  $\square$

Consequently,  $\mathcal{P}_k \cap \mathcal{P}_k^r$  is nothing else but  $\mathcal{P}_k \cap \mathcal{P}'_k$ .

In closing we take the opportunity to recapitulate the situation.

**Remark 6.3.** As noted earlier a fuller, more complete classification of positive maps that goes beyond mere CP and co-CP maps is needed if the theory is to afford the effective construction of non-trivial quantum correlations. In particular, a clearer understanding of the behavior of  $k$ -(co)positivity and  $k$ -decomposability seems to be necessary for a modern theory of positive maps on operator algebras. Even in the finite-dimensional case the situation is still far from clear, with many challenges remaining. In fact, on passing from CP maps to the study of  $k$ -positive maps, many subtleties that seem to be suppressed in the context of CP maps emerge to further complicate the situation. (Compare for example Størmer's result (loc.cit. 1.1) with Theorem 3.6.) Whilst we do not claim to achieve a complete description of  $k$ -positive and  $k$ -decomposable maps in this paper, we do feel that the results pertaining to the transposed cone in Sections 5 and 6 combined with the connection to the Tomita–Takesaki theory achieved in Section 4 succeed in presenting an alternative theoretical framework within which to attempt a more complete classification of the class of unital positive maps observing Detailed Balance II (see [21]). Part of the value of this framework is of course that it brings the full force of the Tomita–Takesaki theory to bear on the problem. Additionally, the description of  $k$ -decomposability at the Hilbert-space level is an alternative approach to characterizing decomposable maps which exploits the powerful duality between operator algebras and the geometry of state-spaces. As such it therefore represents a potentially important stepping stone in the development of the theory.

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